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EXTREME POINTS IN A CLASS OF POLYNOMIALS HAVING UNIVALENT SEQUENTIAL LIMITS

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Abstract. This paper concerns a class $\mathscr{P}_n$ (defined below) of polynomials of degree less than or equal to n having the properties: each polynomial which is univalent in the unit disk and of degree n or less is in $\mathscr{P}_n$ and if $\{P_{n_k}\}_{k=1}^{\infty}$ is a sequence of polynomials such that $P_{n_k} \in \mathscr{P}_{n_k}$ and $\lim_{k\to\infty} P_{n_k} = f$ (uniformly on compact subsets of the unit disk) then f is univalent. The approach is to study the extreme points in $\mathscr{P}_n$ ($P \in \mathscr{P}_n$ is extreme if P is not a proper convex combination of two distinct elements of $\mathscr{P}_n$). Theorem 3 shows that if $P \in \mathscr{P}_n$ is extreme then $((n+1)/n)P(z)-(1/n)zP'(z)$ is univalent and Theorem 6 gives a geometric condition on the image of the boundary of the disk under this mapping in order that P be extreme. Theorem 10 states that the collection of polynomials univalent in the unit disk and having the property $P(z)=z+a_2z^2+\cdots+a_nz^n$, $a_n=1/n$, are dense in the class S of normalized univalent functions. These polynomials have the very striking geometric property that the tangent line to the curve $P(e^{i\theta})$, $0\leqq\theta\leqq 2\pi$, turns at a constant rate (between cusps) as θ varies.

For $n \geqq 1$, let $\mathscr{P}_n$ be the collection of polynomials of degree less than or equal to n of the form $P(z)=z+a_2z^2+\cdots+a_nz^n$ such that the equations

$$(1)\quad \begin{aligned} \frac{\Delta_k P(z)}{z} &= \frac{P(ze^{ik\pi/(n+1)})-P(ze^{-ik\pi/(n+1)})}{z(e^{ik\pi/(n+1)}-e^{-ik\pi/(n+1)})} \\ &= 1+\sum_{j=2}^{n} a_j \frac{\sin kj\pi/(n+1)}{\sin k\pi/(n+1)} z^{j-1} = 0, \qquad k = 1, 2, \ldots, n, \end{aligned}$$

have no roots in $|z|<1$. Since P is univalent in $|z|<1$ if and only if for $0<\theta<\pi/2$ the equation

$$O = 1+\sum_{j=2}^{n} a_j \frac{\sin j\theta}{\sin\theta} z^{j-1}$$

has no roots in $|z|<1$, [3], $\mathscr{P}_n$ contains the collection U_n of all univalent polynomials of degree n or less which are appropriately normalized.

We say that $P \in \mathscr{P}_n$ is an extreme point of $\mathscr{P}_n$ if there do not exist P_1 and P_2 in $\mathscr{P}_n$, $P_1 \neq P_2$, such that $P=tP_1+(1-t)P_2$ where $0<t<1$. We will show below that $\bigcup_{n=1}^{\infty}\mathscr{P}_n$ is a normal family and it is then easy to see that, for each n, $\mathscr{P}_n$ is a compact

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subset of a locally convex linear topological space. As in [2], the Kreĭn-Milman theorem [4] then applies and $\mathscr{P}_n$ is contained in the closure of the convex hull of its extreme points. Further, any continuous linear functional on $\mathscr{P}_n$ assumes its maximum real part and maximum modulus on the set of extreme points.

It is clear that $P \in \mathscr{P}_n$ is an extreme point if and only if for each real α, $e^{-i\alpha}P(ze^{i\alpha})$ is extreme. Hence in attempting to characterize the extreme points of $\mathscr{P}_n$ we may assume $P(z)=z+a_2z^2+\cdots+a_nz^n$ where $a_n \geqq 0$.

THEOREM 1. *If $P(z)=z+\sum_{j=2}^n a_jz^j$ is an extreme point of $\mathscr{P}_n$ such that $a_n\geqq 0$ then $a_{j+1}=\bar{a}_{n-j}$, $0\leqq j\leqq n-1$.*

Proof. Note that

$$\begin{aligned}\sin k(j+1)\pi/(n+1) &= (-1)^{k-1}\sin[k\pi-k(j+1)\pi/(n+1)]\\ &= (-1)^{k-1}\sin k(n-j)\pi/(n+1)\end{aligned}$$

so that $a_n\leqq 1$ with equality only if all the roots of the equations $\Delta_kP(z)/z=0$ lie on $|z|=1$ and in this case $a_{j+1}=\bar{a}_{n-j}$ [1]. Thus we need only show $a_n=1$.

Let

$$\hat{P}(z) = z^{n+1}\overline{P(1/\bar{z})} = \sum_{j=1}^n \bar{a}_{n-j+1}z^j$$

and observe that $\Delta_k(\hat{P})=(-1)^{k-1}(\Delta_kP)^\wedge$. Assume $1>a_n$ and define $Q(z)=(1+a_n)^{-1}[P(z)+\hat{P}(z)]$, $R(z)=(1-a_n)^{-1}[P(z)-\hat{P}(z)]$. We now show Q, $R\in\mathscr{P}_n$. For any polynomial S of degree less than or equal to n,

$$\hat{S}(e^{i\theta}) = e^{i(n+1)\theta}\overline{S(e^{i\theta})}$$

so $\Delta_k\hat{P}/\Delta_kP=(-1)^{k-1}(\Delta_kP)^\wedge/\Delta_kP$ is analytic in a neighborhood of the closed disk. But

$$\begin{aligned}|\Delta_k\hat{P}(z)/\Delta_kP(z)| &= 1 \quad \text{on } |z|=1,\\ &= a_n \quad \text{at } z=0,\end{aligned}$$

so $|\Delta_k\hat{P}/\Delta_kP|<1$ in $|z|<1$. This means $\Delta_kQ\neq 0\neq\Delta_kR$ in $0<|z|<1$ so Q, $R\in\mathscr{P}_n$. However $P=(1+a_n)/2Q+(1-a_n)/2R$ and P is extreme so $Q=R$. This implies $\hat{P}=a_nP$ and equating nth coefficients, $a_n^2=1$, $a_n=1$ which is a contradiction. This completes the proof of Theorem 1.

Now consider the polynomials

$$Q_p(z;n) = \sum_{j=1}^n \frac{\sin jp\pi/(n+1)}{\sin p\pi/(n+1)}z^j = \frac{z(1-(-1)^pz^{n+1})}{1-2z\cos p\pi/(n+1)+z^2}, \qquad 1\leqq p\leqq n. \tag{2}$$

Since

$$\Delta_kQ_p(z;n) = \frac{z(1-(-1)^{k+p}z^{n+1})(1-z^2)}{(1-2ze^{ik\pi/(n+1)}\cos p\pi/(n+1)+z^2e^{2ik\pi/(n+1)})(1-2ze^{-ik\pi/(n+1)}\cos p\pi/(n+1)+z^2e^{-2ik\pi/(n+1)})}, \tag{3}$$

$1 \leqq k \leqq n$, $1 \leqq p \leqq n$, each have $n-1$ zeros on $|z|=1$, we conclude $Q_p(z; n) \in \mathscr{P}_n$. Also, we see that

$$\begin{aligned} \Delta_k Q_p(1; n) &= 0 && \text{if } 1 \leqq k \leqq n, k \neq p, \\ &= (n+1)/(2\sin^2 p\pi/n+1) && \text{if } k = p, \end{aligned} \tag{4}$$

so the polynomials $Q_p(z; n)$ are linearly independent.

THEOREM 2. *If* $P(z)=z+a_2z^2+\cdots+a_nz^n \in \mathscr{P}_n$ *is such that* $a_n=1$, *then* $P(z) = \sum_{p=1}^n \alpha_p Q_p(z; n)$ *where* α_p *is real when p is odd and pure imaginary when p is even. Further* $\sum_{p \text{ odd}} \alpha_p = 1$ *and* $\sum_{p \text{ even}} \alpha_p = 0$.

Proof. Since the $Q_p(z; n)$ are linearly independent, we may write

$$P(z) = \sum_{p=1}^n \alpha_p Q_p(z; n).$$

Then $\Delta_p P(1) = \alpha_p \Delta_p Q_p(1; n)$ by (4) and we have $\alpha_p = \Delta_p P(1)(2\sin^2 p\pi/(n+1))/(n+1)$.

As remarked before, $a_n=1$ implies the coefficient relation $a_{j+1}=\bar{a}_{n-j}$ when $P \in \mathscr{P}_n$ so

$$\Delta_p P(1) = 1+\frac{\sin 2p\pi/(n+1)}{\sin p\pi/(n+1)} a_2+\cdots+(-1)^{p-1}\frac{\sin 2p\pi/(n+1)}{\sin p\pi/(n+1)} \bar{a}_2+(-1)^{p-1}$$

which is real if p is odd and pure imaginary if p is even. The rest of the theorem follows from the normalization of P.

From Theorem 1, we easily obtain the following corollary.

COROLLARY 1. *If* $P \in \mathscr{P}_n$ *is extreme and* $a_n \geqq 0$ *then* $P(z)=\sum_{p=1}^n \alpha_p Q_p(z; n)$ *where* α_p *is real when p is odd and pure imaginary when p is even. Further* $\sum_{p \text{ odd}} \alpha_p = 1$ *and* $\sum_{p \text{ even}} \alpha_p = 0$.

In [6, p. 496] the polynomials $P(z; n, j)$ defined by

$$P(z; n, j) = \sum_{k=1}^n \frac{n-k+1}{n} \frac{\sin kj\pi/(n+1)}{\sin j\pi/(n+1)} z^k$$

were introduced and shown to be univalent. These polynomials are related to the polynomials $Q_p(z; n)$ by the equation $P(z; n, p) = ((n+1)/n)Q_p(z; n) - (1/n)zQ_p'(z; n)$. If $P \in \mathscr{P}_n$, let $P^*(z) = ((n+1)/n)P(z) - (1/n)zP'(z)$. We show below that if $P \in \mathscr{P}_n$ and $a_n=1$ then P^* is univalent in the disk. We require the following lemma.

LEMMA 1. *If* $P(z)=\sum_{j=1}^n a_j z^j \in \mathscr{P}_n$ *then*

$$P^*(z) = \sum_{j=1}^n \frac{n-j+1}{n} a_j z^j \in \mathscr{P}_n.$$

Proof. Observe that

$$\Delta_k P^*(z) = \frac{n+1}{n} \Delta_k P(z) - \frac{1}{n} \Delta_k[zP'(z)] = \frac{n+1}{n} \Delta_k P(z) - \frac{1}{n} z(\Delta_k P)'(z).$$

Since $\Delta_k P(z)=z\prod_{j=1}^{n-1}(1-z/z_j)$, $|z_j|\geqq 1$, we have

$$\operatorname{Re}\left[\frac{z(\Delta_k P)'(z)}{\Delta_k P(z)}\right] = \operatorname{Re}\left[1-\sum_{j=1}^{n-1}\frac{z/z_j}{1-z/z_j}\right] \leqq 1+\frac{n-1}{2} = \frac{n+1}{2}.$$

Hence

$$\left|\frac{\Delta_k P^*(z)}{z}\right| = \left|\frac{\Delta_k P(z)}{nz}\right|\left|n+1-\frac{z(\Delta_k P)'(z)}{\Delta_k P(z)}\right| \geqq \frac{|\Delta_k P(z)|}{n|z|}\cdot\frac{n+1}{2} \neq 0$$

when $|z|<1$.

REMARK. It is also clear in the above proof that $\Delta_k P^*(z_0)=0$ for some z_0 on $|z|=1$ if and only if $\Delta_k P(z)$ has a double zero at $z=z_0$.

THEOREM 3. *If $P(z)=\sum_{j=1}^{n} a_j z^j \in \mathscr{P}_n$ and $|a_n|=1$, then P^* is univalent in $|z|<1$.*

Proof. We may assume $a_n=1$. Using the coefficient relation $a_{j+1}=\bar{a}_{n-j}$ and proceeding as in [6, pp. 497–498] we find $e^{i\theta}P^{*\prime}(e^{i\theta})=e^{i(n+1)\theta/2}R(\theta)$ where R is real valued and $\operatorname{Re}[e^{i\theta}P^{*\prime\prime}(e^{i\theta})/P^{*\prime}(e^{i\theta})+1]=(n+1)/2$ when $P^{*\prime}(e^{i\theta})\neq 0$. That is, the tangent line to the curve $P^*(e^{i\theta})$, $0\leqq\theta\leqq 2\pi$, turns at a constant rate in a counterclockwise direction as θ increases except at the cusps where it reverses direction.

We wish to show that for each θ, $0\leqq\theta<\pi/2$, the polynomial

$$S(z,\theta) = (P^*(ze^{i\theta})-P^*(ze^{-i\theta}))/z(e^{i\theta}-e^{-i\theta}) \qquad (= P^{*\prime}(z) \text{ if } \theta = 0)$$

has no zeros in $|z|<1$. We first show $P^{*\prime}(z)\neq 0$ in $|z|<1$ so suppose $S(z_0,0)=0$ for some z_0, $|z_0|<1$. Since for each θ, $S(z,\theta)$ is a polynomial and the zeros of a polynomial vary continuously with the coefficients there is a continuous function $z(\theta)$ such that $S(z(\theta),\theta)=0$, $0\leqq\theta\leqq\pi/n+1$, $z(0)=z_0$. By Lemma 1, $S(z,\pi/(n+1))\neq 0$ in $|z|<1$ so $|z(\pi/(n+1))|\geqq 1$. Therefore $|z(\phi)|=1$ for some ϕ, $0<\phi\leqq\pi/(n+1)$. For each θ, $0<\theta<\phi$, one can find tangent lines $L(\theta)$, $M(\theta)$ to the closed curve $\gamma(\theta)=\{P(z(\theta)e^{i\psi}):(-\theta\leqq\psi\leqq\theta)\}$ such that $\gamma(\theta)$ is contained between $L(\theta)$ and $M(\theta)$ and so that L and M vary continuously with θ (for example choose L and M parallel to the tangent to $P(z(\theta)e^{i\psi})$ $(|\theta-\psi|<\varepsilon)$ where ε is small). Hence one obtains parallel tangents $L(\phi)$, $M(\phi)$ to the closed curve $\gamma(\phi)$ at the points $P(z(\phi)\exp(i\psi_1))$ and $P(z(\phi)\exp(i\psi_2))$ where $0<\psi_2-\psi_1<2\phi\leqq 2\pi/(n+1)$. But the tangent line turns at the constant rate $(n+1)/2$ on $|z|=1$ so L and M parallel implies $\psi_2-\psi_1=2k\pi/(n+1)\geqq 2\pi/(n+1)$ which is a contradiction. We remark that it seems necessary to find $L(\phi)$ and $M(\phi)$ as above to avoid the problem of cusps on the image of $|z|=1$.

Now suppose $S(z,\theta)=0$ for some θ and z, $0<\theta<\pi/2$, $|z|<1$. Let r be a minimum such that for some z_0 and θ_0, $r=|z_0|$ and $S(z_0,\theta_0)=0$. As before, there is a continuous function $z(\theta)$, $0\leqq|\theta-\theta_0|\leqq\pi/(n+1)$, such that $S(z(\theta),\theta)=0$ and $z(\theta_0)=z_0$. Again using Lemma 1, we conclude there are ϕ_1 and ϕ_2 such that $-\pi/(n+1)<\phi_2$

$-\theta_0 < 0 < \phi_1 - \theta_0 < \pi/(n+1)$ and $|z(\phi_1)| = |z(\phi_2)| = 1$. If for some continuous branch of the argument, we have

$$\psi_1 = \arg(z(\phi_1)\exp(i\phi_1)) > \arg(z(\phi_2)\exp(i\phi_2)) = \psi_2$$

and

$$\psi_3 = \arg(z(\phi_2)\exp(-i\phi_2)) > \arg(z(\phi_1)\exp(-i\phi_1)) = \psi_4$$

we may proceed as in the proof that $P^{*\prime}(z) \neq 0$ in $|z| < 1$ to show that there exist θ_1 and θ_2 satisfying $\psi_1 > \theta_1 > \psi_2$, $\psi_3 > \theta_2 > \psi_4$ and $\theta_1 - \theta_2 = 2j\pi/(n+1) > 0$. But $\psi_1 - \psi_4 = 2\phi_1 > \theta_1 - \theta_2 > \psi_2 - \psi_3 = 2\phi_2$ so $\phi_1 > j\pi/(n+1) > \phi_2$ and $S(z(j\pi/(n+1)), j\pi/(n+1)) = 0$ contradicting Lemma 1.

Let γ be the curve $P^*(z_0 e^{i\theta})$, $0 \leqq \theta \leqq 2\pi$. The curve γ has a common tangent line where $\theta = \theta_0$ and $\theta = -\theta_0$ by the way in which z_0 and θ_0 were chosen. Since $zP^{*\prime}(z)$ is in the direction of the outward normal when $P^{*\prime} \neq 0$ we must have

$$\frac{z_0 \exp(i\theta_0) P^{*\prime}(z_0 \exp(i\theta_0))}{z_0 \exp(-i\theta_0) P^{*\prime}(z_0 \exp(-i\theta_0))} < 0.$$

Using the fact that $P^*(z(\theta)e^{i\theta}) - P^*(z(\theta)e^{-i\theta}) = 0$, we find

$$\begin{aligned} \left| \frac{d \arg z(\theta)}{d\theta} \right| &= \left| \operatorname{Im} \frac{d \log z(\theta)}{d\theta} \right| \\ &= \left| \left(1 + \frac{z(\theta)e^{i\theta}P^{*\prime}(z(\theta)e^{i\theta})}{z(\theta)e^{-i\theta}P^{*\prime}(z(\theta)e^{-i\theta})}\right) \Big/ \left(1 - \frac{z(\theta)e^{i\theta}P^{*\prime}(z(\theta)e^{i\theta})}{z(\theta)e^{-i\theta}P^{*\prime}(z(\theta)e^{-i\theta})}\right) \right| \\ &< 1 \end{aligned}$$

when $\theta = \theta_0$. This means that if ϕ_1 and ϕ_2 are sufficiently near θ_0 and such that $\phi_2 < \theta_0 < \phi_1$ and $|z(\phi_1)| = |z(\phi_2)|$ then $\arg(z(\phi_2)\exp(i\phi_2)) < \arg(z(\phi_1)\exp(i\phi_1))$ and $\arg(z(\phi_2)\exp(-i\phi_2)) > \arg(z(\phi_1)\exp(-i\phi_1))$. Therefore to complete the proof of the theorem we need only show that if $1 \geqq |z(\phi_2)| = |z(\phi_1)|$ and either

$$\arg(z(\phi_2)\exp(i\phi_2)) = \arg(z(\phi_1)\exp(i\phi_1))$$

or

$$\arg(z(\phi_2)\exp(-i\phi_2)) = \arg(z(\phi_1)\exp(-i\phi_1))$$

we obtain a contradiction to $P^* \in \mathscr{P}_n$. We have $0 < \phi_1 - \phi_2 < \pi/(n+1)$ and by changing notation there are z and $\theta = \phi_1 - \phi_2$ such that $P^*(ze^{i\theta}) = P^*(ze^{-i\theta})$, $0 < \theta < \pi/(n+1)$. The proof now proceeds as in the proof that $P^{*\prime} \neq 0$ in $|z| < 1$.

Corollary 2. *If $P \in \mathscr{P}_n$ is an extreme point then $((n+1)/n)P(z) - (1/n)zP'(z)$ is univalent in $|z| < 1$.*

We have the following converse to Theorem 3.

Theorem 4. *If*

$$Q(z) = \sum_{j=1}^{n} \frac{n-j+1}{n} a_j z^j, \qquad a_1 = 1 = a_n,$$

and $Q(z)$ is univalent in $|z| < 1$ then $P(z) = \sum_{j=1}^{n} a_j z^j \in \mathscr{P}_n$.

Proof. As shown in [1], we must have $a_{j+1}=\bar{a}_{n-j}$ and it then follows that

$$\text{Re}\,[\Delta_k zP'(z)/\Delta_k P(z)] = \text{Re}\,[z(\Delta_k P)'(z)/\Delta_k P(z)] = (n+1)/2$$

when $|z|=1$, $\Delta_k P(z)\neq 0$.

Suppose $\Delta_k P(z)=0$ for some z, $|z|<1$. Then $w=\Delta_k P'(z)/\Delta_k P(z)=n+1$ for some z, $|z|<1$ for w assumes every value in a neighborhood of ∞ and therefore every value not on the line Re $w=(n+1)/2$. But $\Delta_k Q(z)=((n+1)/n)\Delta_k P(z)-(1/n)\Delta_k zP'(z)=0$ when $w=n+1$ which contradicts the univalence of Q. This proves $\Delta_k P(z)\neq 0$ when $|z|<1$, $1\leqq k\leqq n$ so $P\in\mathscr{P}_n$ and the proof is complete.

Now suppose $P(z)=\sum_{j=1}^n a_j z^j\in\mathscr{P}_n$. If $a_n=1$, then by Theorem 3, P^* is univalent in $|z|<1$ so $|a_j|<n(3j)/(n-j+1)<6nj/(n+1)<6j$ if $j\leqq(n+1)/2$. Using the coefficient relation, $|a_j|<6j$ for all j. If $0\leqq a_n<1$ then as shown in the proof of Theorem 1, P is a convex combination of members of $\mathscr{P}_n$ having nth coefficient ± 1. Therefore, in any case $|a_j|<6j$. Hence for $P\in\mathscr{P}_n$, $|P(z)|<6\sum_{j=1}^n j|z|^j<6|z|/(1-|z|)^2$ so the family $\bigcup_{n=1}^\infty \mathscr{P}_n$ is locally uniformly bounded and is therefore a normal family.

THEOREM 5. *Suppose $P_{n_k}\in\mathscr{P}_{n_k}$ and that $P_{n_k}\to f$ as $k\to\infty$. Then f is univalent in $|z|<1$.*

Proof. Suppose f is not univalent in $|z|<1$. Then there exist θ, z such that $0<|z|<1$, $0<\theta<\pi/2$ and $f(ze^{i\theta})=f(ze^{-i\theta})$. In fact there exists $r<1$ and a closed interval $I=[\theta_1,\theta_2]$ such that the equation $f(ze^{i\theta})=f(ze^{-i\theta})$ has a solution in $D_r=\{z : 0<|z|<r\}$ for each $\theta\in I$. For fixed $\theta\in I$, there exists k such that if $l>k$ then $P_{n_l}(ze^{i\theta})=P_{n_l}(ze^{-i\theta})$ has a solution in $D_{(1+r)/2}$. Let $I_k=\{\theta\in I : P_{n_l}(ze^{i\theta})=P_{n_l}(ze^{-i\theta})$ has a solution in $D_{(1+r)/2}$ for all $l>k\}$. Then $\bigcup_{k=1}^\infty I_k=I$ so by Baires' theorem [7, p. 76] some I_k contains an interval. That is $l>k_0$ implies $P_{n_l}(ze^{i\theta})=P_{n_l}(ze^{-i\theta})$ has a solution in $D_{(1+r)/2}$ for all θ in some fixed interval. This contradicts the definition of $\mathscr{P}_n$ and completes the proof.

We now wish to obtain some geometric properties of the univalent polynomial P^* associated with extreme points $P\in\mathscr{P}_n$. Note that a double zero of $\Delta_k P(z)$ is a zero of $\Delta_k P^*$. Letting $\gamma=\{P^*(e^{i\theta}) : 0\leqq\theta<2\pi\}$ we see that a double root of $\Delta_k P$ on $|z|=1$ corresponds to a point of self-tangency of γ and conversely. Further, we have the following lemma.

LEMMA 2. *If $P\in\mathscr{P}_n$ satisfies $a_n=1$ then $\Delta_k P$ cannot have a zero of multiplicity greater than* 2.

Proof. Assume $\Delta_k P$ has a zero of multiplicity greater than 2. Then $P^*(ze^{ik\pi/(n+1)})=P^*(ze^{-ik\pi/(n+1)})$ and $ze^{ik\pi/(n+1)}P^{*\prime}(ze^{ik\pi/(n+1)})=ze^{-ik\pi/(n+1)}P^{*\prime}(ze^{-ik\pi/(n+1)})$ for some z, $|z|=1$. If $P^{*\prime}(ze^{ik\pi/(n+1)})=0=P^{*\prime}(ze^{-ik\pi/(n+1)})$ then it is clear that the images under the mapping P^* of small sectors of sufficiently large opening inside the unit circle with vertices at $ze^{ik\pi/(n+1)}$ and $ze^{-ik\pi/(n+1)}$ will overlap contradicting the univalence of Q. Hence the image of $|z|=1$ has a common tangent at the two points under consideration and as seen previously this implies

$$\frac{ze^{ik\pi/(n+1)}P^{*\prime}(ze^{ik\pi/(n+1)})}{ze^{-ik\pi/(n+1)}P^{*\prime}(ze^{-ik\pi/(n+1)})} < 0.$$

This is a contradiction which completes the proof.

THEOREM 6. *If* $P\in\mathscr{P}_n$ $(n>2)$ *is an extreme point then the curve* $\gamma=P^*(e^{i\theta}) : 0\leqq\theta\leqq 2\pi$ *has* $n-2$ *points of self-tangency. Further, if* $P^*(\exp(i\theta_2))=P^*(\exp(i\theta_1))$ *then* $\theta_2-\theta_1=2k\pi/(n+1)$ *for some integer* k.

Proof. The last assertion in the theorem follows easily from the fact that $\operatorname{Re}[e^{i\theta}P^{*\prime\prime}(e^{i\theta})/P^{*\prime}(e^{i\theta})+1]=(n+1)/2$ when $P^{*\prime}(e^{i\theta})\neq 0$ so the tangent line to γ turns at a constant rate between cusps as θ varies. Since $P^*(\exp(i\theta_2))=P^*(\exp(i\theta_1))$ implies there is a common tangent line to γ at the above points we must have $((n+1)/2)(\theta_2-\theta_1)=k\pi$ for some integer k.

We now proceed to prove the first part of the theorem.

We may assume $P(z)=\sum_{j=1}^{n} a_jz^j$, $a_n=1$. Note that $\Delta_{k+1}P(z)=-\Delta_{n-k}P(-z)$ so it is sufficient to consider $k\leqq(n+1)/2$. Further, if n is odd, $\Delta_{(n+1)/2}P(z)$ is an even function and we may assume for this polynomial that $0\leqq\arg z<\pi$. Hence we will show that there are $n-2$ values of z such that $\Delta_kP(z)$ has a double zero on $|z|=1$ for some k satisfying $1\leqq k<(n+1)/2$ or $k=(n+1)/2$ and $0\leqq\arg z<\pi$. Suppose this is not the case. We wish to construct

$$\begin{aligned} R(z) &= \sum_{n\geqq k>1;\,k\text{ odd}} \alpha_k(Q_k(z;n)-Q_1(z;n))+i\sum_{n\geqq k>2;\,k\text{ even}}\beta_k(Q_k(z;n)-Q_2(z;n)) \\ &= \sum_{j=2}^{n-1} b_jz^j \end{aligned}$$

where α_k and β_k are real and $Q_k(z;n)$ is given by (2) so that $P(z)+tR(z)\in\mathscr{P}_n$ and $P(z)-tR(z)\in\mathscr{P}_n$ for some $t>0$, $R\not\equiv 0$. Then $P=\frac{1}{2}(P+tR)+\frac{1}{2}(P-tR)$ and P is not extreme (R must have the above form in order for $P+tR$ and $P-tR$ to satisfy the coefficient relation).

We wish to obtain $n-2$ real linear equations in the $n-2$ unknowns α_k, β_k. For each double zero $e^{i\theta}$ of $\Delta_kP(z)$ restricted as discussed above, we obtain an equation by setting $i^{k-1}e^{-i(n+1)\theta/2}\Delta_kR(e^{i\theta})=0$. The coefficient relation in $Q_k(z;n)$ implies that the α_k and β_k have real coefficients in these equations. Suppose l equations are obtained in this way. The remaining equations are obtained by setting $e^{-ij\pi/2}R(e^{ij\pi/(n+1)})=K$, $j=1,2,\ldots,n-l-2$, where $K=0$ if the determinant of coefficients is 0 and $K=1$ otherwise. Thus in any case, there is a choice of the α_k and β_k such that $\Delta_kR(z)=0$ when $\Delta_kP(z)$ has a double zero and $R(z)\not\equiv 0$.

Let k be fixed and consider $S(\theta)=\Delta_kP(e^{i\theta})/\Delta_kR(e^{i\theta})$. Suppose $\Delta_kP(e^{i\theta})$ and $\Delta_kR(e^{i\theta})$ have p common zeros. Then $S(\theta)$ has $n-1-p$ simple zeros and therefore changes sign at each of these zeros. Hence there exists $t_k>0$ such that $S(\theta)$ assumes the values t_k and $-t_k$, $n-1-p$ times. This means that all the zeros of

$\Delta_k P(z)+t\Delta_k R(z)$ and $\Delta_k P(z)-t\Delta_k R(z)$ lie on $|z|=1$ when $t\leqq t_k$. Setting $t=\min_{1\leqq k\leqq n} t_k$ the proof is now complete.

EXAMPLES. $n=1$. $\mathscr{P}_1=\{z\}$.

$n=2$. Theorem 2 implies that the extreme points of $\mathscr{P}_2$ are rotations of $z+z^2$.

$n=3$. Theorem 2 implies that the extreme points of $\mathscr{P}_3$ are rotations of polynomials of the form $P(z)=z+a_2z^2+z^3$ where a_2 is real. Clearly we may assume $a_2\geqq 0$. Theorem 6 implies that one of the polynomials $\Delta_1 P(z)=z+\sqrt{2}a_2z^2+z^3$ or $\Delta_2 P(z)=z-z^3$ has a double zero on $|z|=1$. It follows that $a_2=\sqrt{2}$ so all extreme points of $\mathscr{P}_3$ are rotations of $z+\sqrt{2}z^2+z^3$.

$n=4$. Theorem 2 implies that the extreme points of $\mathscr{P}_4$ are rotations of polynomials of the form $P(z)=z+a_2z^2+\bar{a}_2z^3+z^4$. Theorem 6 implies that

$$1+2a_2\cos(\pi/5)z+2\bar{a}_2\cos(\pi/5)z^2+z^3$$

and

$$1+2a_2\cos(2\pi/5)z-2\bar{a}_2\cos(2\pi/5)z^2-z^3$$

each have a double zero on $|z|=1$. Applying this to P^*, each of the polynomials,

$$\begin{gathered}1+\tfrac{3}{2}a_2\cos(\pi/5)z+\bar{a}_2\cos(\pi/5)z^2+\tfrac{1}{4}z^3 \quad\text{and}\\ 1+\tfrac{3}{2}a_2\cos(2\pi/5)z-\bar{a}_2\cos(2\pi/5)z^2-\tfrac{1}{4}z^3\end{gathered} \tag{5}$$

has exactly one zero on $|z|=1$. By Cohn's rule [1] and [5, p. 149], if $f(z)=c_0+c_1z+\cdots+c_kz^k$ satisfies $|c_0|>|c_k|$ then

$$f^*(z) = \bar{c}_0 f(z)-c_kz^k\overline{f(1/\bar{z})}$$

has the same zeros as f on $|z|=1$ and the same number of zeros as f in $|z|<1$. Applying Cohn's rule twice to each of the polynomials (5) leads to the linear polynomials

$$\left(\frac{12\cos 2\pi/5}{\bar{a}_2}-\frac{2\bar{a}_2}{a_2}\right)z+\frac{36\cos^2 2\pi/5}{|a_2|^2}-1$$

and

$$\left(\frac{12\cos \pi/5}{\bar{a}_2}+\frac{2\bar{a}_2}{a_2}\right)z+\frac{36\cos^2 \pi/5}{|a_2|^2}-1$$

(we have used the fact that $\cos 2\pi/5=(\sqrt{5}-1)/4$ and $\cos\pi/5=(\sqrt{5}+1)/4$ so $\cos(2\pi/5)\cos\pi/5=1/4$) each of which has a zero on $|z|=1$. This fact yields the equations

$$\begin{gathered}|a_2|^4-16\operatorname{Re} a_2^3\cos 2\pi/5+72|a_2|^2\cos^2 2\pi/5-432\cos^4 2\pi/5 = 0,\\ |a_2|^4+16\operatorname{Re} a_2^3\cos \pi/5+72|a_2|^2\cos^2 \pi/5-432\cos^4 \pi/5 = 0.\end{gathered} \tag{6}$$

Eliminating $\operatorname{Re} a_2^3$ from the equations (6) we obtain $|a_2|^4+18|a_2|^2-54=0$ so $|a_2|^2=3\sqrt{15}-9$. Substitution into either equation in (6) then yields

$$\cos(3\arg a_2) = \tfrac{3}{16}\sqrt{(9+5\sqrt{15})}. \tag{7}$$

If we choose a value of arg a_2 to satisfy (7) and choose $|a_2|$ so the equations (6) are satisfied then the extreme points in $\mathscr{P}_4$ are rotations of the polynomials $z+a_2z^2+\bar{a}_2z^3+z^4$ and $z+\bar{a}_2z^2+a_2z^3+z^4$.

$n=5$. The extreme points of $\mathscr{P}_5$ are rotations of polynomials of the form $P(z)=z+(a+b_i)z^2+cz^3+(a-b_i)z^4+z^5$ where a, b and c are real. By considering $-iP(iz)$, $-P(-z)$ and $(P(\bar{z}))^-$ we see that we may assume a, b and c are non-negative. Theorem 6 implies that among the roots of the equations

$$\begin{aligned} 1+\sqrt{3}(a+b_i)z+2cz^2+\sqrt{3}(a-b_i)z^3+z^4 &= 0 \\ 1+(a+b_i)z \qquad -(a-b_i)z^3 \quad -z^4 &= 0 \\ 1 \qquad -cz^2 \qquad +z^4 &= 0 \end{aligned} \tag{8}$$

there must be three double roots on $|z|=1$ (only half of the double roots of the third equation are to be counted). Observe that $(1+e^{i\alpha}z)^2(1+e^{i\beta}z)^2=1+2(e^{i\alpha}+e^{i\beta})z+(e^{2i\alpha}+e^{2i\beta}+4e^{i(\alpha+\beta)})z^2+2(e^{i(2\alpha+\beta)}+e^{i(2\beta+\alpha)})z^3+e^{i2(\alpha+\beta)}z^4$ so the second equation in (8) cannot have two double roots on $|z|=1$. Suppose the first equation has two double roots on $|z|=1$. Then the left hand-side has the form above where $\beta=-\alpha$ or $\beta=\pi-\alpha$. Since $c\geqq 0$, we must have $\beta=-\alpha$ so $b=0$. The second equation in (8) then has roots ± 1 together with the zeros of $1+az+z^2$. Hence in this case the second equation cannot have a double root (the only possibility is -1 and it is either a simple root or a triple root). This means the third equation has a double root so $c=2$ and $a=\sqrt{\frac{8}{3}}$. But

$$z+\sqrt{\tfrac{8}{3}}z^2+2z^3+\sqrt{\tfrac{8}{3}}z^4+z^5 = ((\sqrt{8}+3)/6)Q_1(z;5)+((3-\sqrt{8})/6)Q_5(z;5)$$

and this polynomial is not an extreme point.

Thus we conclude that if P is extreme then each of the equations in (8) has a double root on $|z|=1$. From the third equation, $c=2$. Assume that $e^{i\phi}$ and $e^{i\theta}$ are double roots of the first and second equations respectively. We obtain the system

$$\begin{aligned} \cos 2\phi+\sqrt{3}a\cos\phi+\sqrt{3}b\sin\phi+2 &= 0, \\ 2\sin 2\phi+\sqrt{3}a\sin\phi-\sqrt{3}b\cos\phi &= 0, \\ \sin 2\theta+a\sin\theta-b\cos\theta &= 0, \\ 2\cos 2\theta+a\cos\theta+b\sin\theta &= 0. \end{aligned} \tag{9}$$

Solving for a and b in terms of θ and ϕ we find

$$\begin{aligned} a &= -2\cos^3\theta = (1/2\sqrt{3})(\cos 3\phi-7\cos\phi), \\ b &= 2\sin^3\theta = (1/2\sqrt{3})(\sin 3\phi-\sin\phi). \end{aligned} \tag{10}$$

From equations (10),

$$a^2+b^2 = \tfrac{1}{3}(8+4\cos 2\phi-3\cos^2 2\phi) \tag{11}$$

so $a^2+b^2<28/9$.

Now let $\theta=\theta(\phi)$ satisfy $-2\cos^3\theta=(1/2\sqrt{3})(\cos 3\phi-7\cos\phi)$ and set $g(\phi)=2\sin^3\theta-(1/2\sqrt{3})(\sin 3\phi-\sin\phi)$. Since $a>0$ and $b>0$ we have $\pi>\phi$, $\theta>3\pi/4$.

Also $g'(\phi)=(10-12\cos^2\phi)\cos(\phi-\theta)/(2\sqrt{3}\cos\theta)$, $g(\pi)>0$, $g(\phi_0)<0$ and $g(3\pi/4)>0$ where $\cos^2\phi_0=\frac{5}{6}$. Thus we conclude the system (10) has two solutions. Assume the values of ϕ corresponding to these solutions are ϕ_1 and ϕ_2 where $\pi>\phi_1>\phi_0>\phi_2>3\pi/4$. From (11), we find $a^2+b^2=3$ when $\cos 2\phi=1$, $\frac{1}{3}$ while $g(\phi)<0$ when $\cos 2\phi=\frac{1}{3}$ so $3<a^2+b^2<28/9$ for the solution corresponding to ϕ_1 and $a^2+b^2<3$ for the solution corresponding to ϕ_2. Denote the polynomials P corresponding to the solutions of (10) for $\phi=\phi_1, \phi_2$ by P_1 and P_2 respectively and let θ_1 and θ_2 be the corresponding values of θ. Clearly P_1 is an extreme point since it maximizes $|a_2|$ ($P_1\in\mathscr{P}_5$ since there must be an extreme point satisfying $\sqrt{3}\leqq|a_2|$).

We now show P_2 is also an extreme point of $\mathscr{P}_5$. Note that in any of the equations in (8), the roots which do not lie on $|z|=1$ must occur as pairs of roots which are inverse points with respect to $|z|=1$. Hence if two roots vary continuously beginning on $|z|=1$ and ending as inverse points with respect to the circle then they must at some time coincide. The third equation in (8) has its roots on the circle when $c=2$. Setting $a(\theta)=-2\cos^3\theta$ and $b(\theta)=2\sin^3\theta$, $e^{i\theta}$ is a double root of the second equation and the other roots lie on $|z|=1$ for we can never have $a(\alpha)=a(\theta)$ and $b(\alpha)=b(\theta)$ when $\alpha\neq\theta \bmod 2\pi$ (and the roots do lie on $|z|=1$ when $\theta=0$). Now set $a(\phi)=(1/2\sqrt{3})(\cos 3\phi-7\cos\phi)$ and $b(\phi)=(1/2\sqrt{3})(\sin 3\phi-\sin\phi)$. All roots of the first equation in (8) lie on $|z|=1$ when $\phi=\pi$. Further, if $\pi>\phi>3\pi/4$ then $a(\phi)=a(\alpha)$ and $b(\phi)=b(\alpha)$ together imply $\phi=\alpha$. Hence for $\phi=\phi_2$, all roots of the first equation in (8) lie on $|z|=1$. Hence $P_2\in\mathscr{P}_5$.

Now suppose P_2 is not an extreme point. Then P_2 is in the convex hull of the set

$$\{e^{-i\alpha}P_1(e^{i\alpha}z) : \alpha \text{ is real}\} \cup \{e^{i\beta}\overline{P_1(\bar{z}e^{i\beta})} : \beta \text{ is real}\}.$$

Since P_2 has third coefficient 2 and fifth coefficient 1, P_2 is a convex combination of

$$P_1(z),\quad \overline{P_1(\bar{z})},\quad -P_1(-z) \quad\text{and}\quad \overline{-P_1(-\bar{z})}.$$

Therefore, $\sin^3\theta_2$ is a convex combination of $\sin^3\theta_1$ and $-\sin^3\theta_1$. But $\sin\theta_2>\sin\theta_1$ and this is impossible so P_2 is extreme.

For the next theorem, we restrict ourselves to the subclass $\mathscr{R}_n\subset\mathscr{P}_n$ having the property $P\in\mathscr{R}_n$ implies P has real coefficients.

THEOREM 7. *For each $p=1, 2, \ldots, n$, $Q_p(z; n)$ is an extreme point of $\mathscr{R}_n$. Further $\mathscr{R}_n\subset$ co $\{Q_p(z; n)\}_{p=1}^n)$ where* co (A) *is the convex hull of A.*

Proof. Let $P(z)=z+\sum_{j=2}^n a_jz^j\in\mathscr{R}_n$ be extreme. As in the proof of Theorem 1, we may show that if P is extreme then $a_n=\pm 1$. Using Theorem 3 above, Theorem 2 of [6, p. 500] and the fact mentioned previously that $Q_p^*(z; n)=P(z; n, p)$ we have $P(z)=\sum_{p=1}^n \alpha_pQ_p(z; n)$ where $\alpha_p\geqq 0$ and $\sum_{p=1}^n\alpha_p=1$. This proves

$$\mathscr{R}_n \subset \text{co}\,(\{Q_p(z; n)\}_{p=1}^n).$$

Every $Q\in\mathscr{R}_n$ can be written uniquely in the form $Q(z)=\sum_{p=1}^n\alpha_pQ_p(z; n)$, $\alpha_p\geqq 0$, $\sum_{p=1}^n\alpha_p=1$ and the functional J_k defined on $\mathscr{R}_n$ by $J_k(Q)=\alpha_k$ is a continuous linear functional. Clearly $\alpha_k = 1$ is its maximum which is assumed only when $Q(z) = Q_k(z; n)$. This proves $Q_k(z; n)$ is extreme in $\mathscr{R}_n$.

From Theorems 6 and 7 above, it seems likely that no extreme points of $\mathscr{P}_n$ can have all real coefficients when $n \geqq 4$. At least we have the following theorem.

THEOREM 8. *If* $n > 5$ *and* $1 < j < n$ *then*

$$\max_{P \in \mathscr{P}_n} |a_j| > \frac{\sin j\pi/(n+1)}{\sin \pi/(n+1)} = \max_{P \in \mathscr{R}_n} |a_j|$$

where $P(z) = \sum_{j=1}^{n} a_j z^j$.

Proof. The equality

$$\max_{P \in \mathscr{R}_n} |a_j| = \frac{\sin j\pi/(n+1)}{\sin \pi/(n+1)}$$

follows from Theorem 3 and [6, Theorem 3]. From (3) we see that $\Delta_k Q_1(z; n)/z$ has simple zeros except possibly at ± 1 and $\Delta_k[Q_4(z; n) - Q_2(z; n)]/z$ has a simple zero at ± 1 when $\Delta_k Q_1(z; n)/z$ has a double zero, $n > 5$. It then follows by the same argument as used in the proof of Theorem 6 that for t sufficiently small, $Q(z) = Q_1(z; n) + it[Q_4(z; n) - Q_2(z; n)] \in \mathscr{P}_n$ and all zeros of $\Delta_k Q$ are simple zeros, $1 \leqq k \leqq n$. For fixed j, choose p odd so that

$$\left| \frac{\sin jp\pi/(n+1)}{\sin p\pi/(n+1)} \right| < \frac{\sin j\pi/(n+1)}{\sin \pi/(n+1)}.$$

Applying the same argument, it follows that, for sufficiently small s,

$$P(z) = Q_1(z; n) - s(Q_p(z; n) - Q_1(z; n)) + it(Q_4(z; n) - Q_2(z; n))$$

is in $\mathscr{P}_n$. Then $|a_j| \geqq \operatorname{Re} a_j > (\sin j\pi/(n+1))/(\sin \pi/(n+1))$. Actually the conclusion of the theorem holds for $n > 3$ except for the case $n = 5, j = 3$.

THEOREM 9. *Every function* f *in the class* S *of functions univalent in* $|z| < 1$ *and normalized by setting* $f(0) = 0, f'(0) = 1$, *is the limit of polynomials of the form* $P(z) = z + \sum_{j=2}^{n} a_j z^j \in \mathscr{P}_n$ *which satisfy* $a_n = 1$.

Proof. Let $f \in S$. One can obtain a sequence of univalent polynomials by taking appropriate partial sums of $r_k^{-1} f(r_k z)$ where $\{r_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of real numbers such that $\lim_{k \to \infty} r_k = 1$. Let $\{Q_{n_k}\}$ be such a sequence where Q_{n_k} is of degree n_k. Define

$$P_{n_k}(z) = Q_{n_k}(z) + z^{2n_k+1}\overline{Q_{n_k}(1/\bar{z})} = \sum_{j=1}^{2n_k} a_j z^j.$$

By an argument similar to that used to prove Theorem 1, $P_{n_k} \in \mathscr{P}_{2n_k}$ and $a_{2n_k} = 1$. Also $\lim_{k \to \infty} P_{n_k}(z) = f(z)$ so $\{P_{n_k}\}_{k=1}^{\infty}$ is the required sequence.

THEOREM 10. *The univalent polynomials of the form* $z+\sum_{j=2}^{n} a_j z^j$ *which satisfy* $(j+1)a_{j+1}=(n-j)\bar{a}_{n-j}$ [*and thus* $a_n=1/n$] *are dense in the class* S.

Proof. The polynomials $\{P_{n_k}^*\}_{k=1}^{\infty}$ have the same limit as $\{P_{n_k}\}_{k=1}^{\infty}$ above.

We observe that in Theorem 9, if $f \in S$ has real coefficients then the sequence $\{P_{n_k}^*\}_{k=1}^{\infty}$ has real coefficients and we have a new proof of the Bieberbach conjecture for functions having real coefficients.

Let $\mathscr{D}_n$ be the class of polynomials of degree n which are univalent in $|z|<1$ and of the form $z+\sum_{j=2}^{n} a_j z^j$, $(j+1)a_{j+1}=(n-j)\bar{a}_{n-j}$. Applying Theorems 2 and 4 above and using the definition of $P(z; n, j)$ in [6] we can represent any $P \in \mathscr{D}_n$ in the form

$$P(z) = P(z; n, 1)+\sum_{j\,\mathrm{odd}} \alpha_j[P(z; n, j)-P(z; n, 1)] + i\sum_{j\,\mathrm{even}} \beta_j[P(z; n, j)-P(z; n, 2)]. \tag{12}$$

Since the tangent line to the curve $P(e^{i\theta})$ $(0 \leqq \theta \leqq 2\pi)$ turns at a constant rate (Re $[e^{i\theta}P''(e^{i\theta})/P'(e^{i\theta})+1]=(n+1)/2$ when $P'(e^{i\theta})\neq 0$) as θ varies the tangent line to the curve is horizontal when θ is an odd multiple of $\pi/(n+1)$ and vertical when θ is an even multiple of $\pi/(n+1)$. Recall that among all polynomials in $\mathscr{D}_n$ having real coefficients, $P(z; n, 1)$ maximizes every coefficient. Also $\Delta_k P(1; n, 1)=0$ when k is odd, $k>1$ (i.e. $P(e^{ik\pi/(n+1)}; n, 1)=P(e^{-ik\pi/(n+1)}; n, 1)$ when k is odd, $k>1$). Note also that for even $j>2$ and odd k, $\Delta_k[P(1; n, j)-P(1; n, 2)]$ is pure imaginary. Hence the effect of adding $it[P(z; n, j)-P(z; n, 2)]$ (where t is real and near 0 and j is even, $j>2$) to $P(z; n, 1)$ is to shift the values at $e^{ik\pi/(n+1)}$ and $e^{-ik\pi/(n+1)}$ apart horizontally. Finally, in the representation (12), for odd $k>1$ we have $\alpha_k \operatorname{Im}[\Delta_k P(1; n, k)]=\operatorname{Im}[\Delta_k P(1)]$ so α_k is negative when $\operatorname{Im} P(e^{-ik\pi/(n+1)}) > \operatorname{Im} P(e^{ik\pi/(n+1)})$. Since the coefficients in $P(z; n, k)-P(z; n, 1)$ are all nonpositive, it would appear that to obtain the maximum modulus for any coefficient, one should choose the β_j to shift the values $P(e^{ik\pi/(n+1)})$ and $P(e^{-ik\pi/(n+1)})$ apart and then choose the α_j negative, the choices being made to satisfy Theorem 6. This discussion leads to the following conjecture.

CONJECTURE. Among all polynomials $P(z)=z+\sum_{j=2}^{n} a_j z^j \in \mathscr{D}_n$, the quantities $|a_j|$, $2\leqq j\leqq n-1$, are all maximized by a single polynomial having the property that in the representation (12), $\alpha_j \leqq 0$ for all odd j, $n \geqq j>1$.

This conjecture implies the Bieberbach conjecture as shown by the following argument. Suppose $P_n(z)$ maximizes $|a_j|$, $2\leqq j\leqq n-1$, in $\mathscr{D}_n$. Let $f(z)=\sum_{j=1}^{\infty} b_j z^j \in S$ and let $Q_{n_k} \in \mathscr{D}_{n_k}$ where $\{Q_{n_k}\}_{k=1}^{\infty}$ is a sequence of polynomials having f as limit. Let $P_n(z)=\sum_{j=1}^{n} a_{j,n} z^j$, $Q_{n_k}(z)=\sum_{j=1}^{n_k} b_{j,n_k} z^j$. Then $|a_{j,n_k}| \geqq |b_{j,n_k}|$ for each k and $2>|a_{2,n_k}|\geqq ((n_k-1)/n_k)\cos \pi/(n_k+1)$. Hence $\lim_{k\to\infty} |a_{2,n_k}|=2$ and any convergent subsequence of $\{P_n\}$ must converge to a Koebe function. After possibly renaming to obtain a convergent subsequence, we therefore have

$$j = \lim_{k\to\infty} |a_{j,n_k}| \geqq \lim_{k\to\infty} |b_{j,n_k}| = |b_j|.$$

REFERENCES

1. D. A. Brannan, *Coefficient regions for univalent polynomials of small degree*, Mathematika **14** (1967), 165–169. MR **36** #3971.

2. Louis Brickman, *Extreme points of the set of univalent functions*, Bull. Amer. Math. Soc. **76** (1970), 372–374. MR **41** #448.

3. J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe*, Ann. Ecole Norm. Sup. (3) **48** (1931), 247–358.

4. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

5. M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Math. Surveys, no. 3, Amer. Math. Soc., Providence, R. I., 1949. MR **11**, 101.

6. T. J. Suffridge, *On univalent polynomials*, J. London Math. Soc. **44** (1969), 496–504. MR **38** #3419.

7. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958. MR **20** #5411.

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